GRAPH THEORY NOTES OF NEW YORK

XIX

Editors:
John W. Kennedy
Louis V. Quintas

(1990)

Graph Theory Notes of New York

This edition ***Graph Theory Notes of New York*** has been retypeset from the original with pagination preserved for index purposes.

Graph Theory Notes of New York publishes short contributions and research announcements in the field of graph theory and its applications. ***Graph Theory Notes*** includes, but is not restricted to, material presented during one-day meetings on graph theory sponsored by the Mathematics Section of the New York Academy of Sciences.

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Published and Distributed by:	The New York Academy of Sciences 2 East 63rd Street New York, NY 10021, U.S.A.
and	Pace University New York, NY 10038, U.S.A.
Typescript:	Using Framemaker™ in a Macintosh™ environment by K_2Q, Pace University, New York, NY 10038, U.S.A.

ISSN 1040-8118

Information for contributors can be found on the inside back cover.

THE NEW YORK ACADEMY OF SCIENCES

GRAPH THEORY NOTES

OF NEW YORK

XIX

(1990)

This issue includes papers from
The Nineteenth
New York Graph Theory Day

sponsored and hosted by
The Mathematics Section of
The New York Academy of Sciences
New York, NY 10021

May 5, 1990

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INTRODUCTORY REMARKS

The Nineteenth Graph Theory Day of the Mathematics Section of The New York Academy of Sciences took place on May 5, 1990 at the Academy Building in New York City. The session were chaired by Professor Fred Buckley (Baruch College, CUNY). At these sessions the featured presentations were:

A Probabilistic/Topological Approach to Graph Isomorphism Testing

by J.L. Gross
Department of Computer Science
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Extremal Problems for Random Walks on Graphs

by P. Winkler
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In addition to the articles by featured speakers and other presenters during Graph Theory Days, *Graph Theory Notes of New York* invites and welcomes contributions from all of our readers. Articles for *Graph Theory Notes of New York* can be sent to the Editors at any time for inclusion in a future issue. We are particularly interested in follow-ups to earlier contributions. These can be in the form of a short article, a remark, or a relevant reference, and can be accommodated in our Developments Section which appears when such material is contributed.

We would like to remind readers that *Graph Theory Notes of New York* are mailed out at no charge. To help us to avoid incurring unnecessary mailing costs, if your address is in error or if you change your address, please inform us. Should anyone you know be interested in receiving *Graph Theory Notes* please pass on a copy of the "subscription request" included at the back of this issue. We particularly welcome efforts by our readers to bring the *Notes* to the attention of their libraries. Librarians also may enter a subscription by returning a copy of the request.

We thank the participants of Graph Theory Day Nineteen and all our readers for their continued support of the *Graph Theory Notes of New York*.

M.L.H./J.W.K./L.V.Q.
Pace University
August, 1990

GRAPH THEORY DAY 19

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Graph Theory Notes of New York XIX, 8–17 (1990)
New York Academy of Sciences.

[GTN XIX:1] A PROBABILISTIC/TOPOLOGICAL APPROACH TO GRAPH ISOMORPHISM TESTING

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Abstract

The “graph isomorphism problem” is to design a practical method for deciding whether two n-vertex graphs are isomorphic. Practicality necessitates a polynomial bound on the number of steps, which excludes brute-force consideration of all $n!$ relabelings, along with several more sophisticated deterministic schemes. It seems conceivable, however, that a carefully partitioned, polynomial-sized random sample of imbedding information about the two graphs might enable us to decide whether they are isomorphic, with a negligible chance of error. This non-deterministic viewpoint is yielding a steady stream of structural, enumerative, and algorithmic results about the system of graph imbeddings. In particular, it is now known that the system of imbeddings completely characterizes the homeomorphism type of a graph; that imbedding distributions of various fundamental families of graphs can be derived with the aid of symmetric representations and other classical algebraic tools; and that the maximum genus of a graph can be calculated by matroid parity or by hill-climbing in polynomial time. The central thrust of continuing attempts at the graph isomorphism problem is to coordinate a complete or nearly complete invariant with a sampling method.

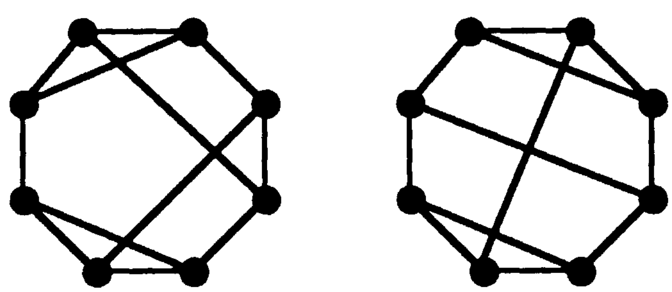

Figure 0: Are they isomorphic?

1. Hierarchy of Topological Invariants of Isomorphism Type

Understanding of this exposition requires reasonable familiarity with non-planar imbeddings. The terminology follows Gross and Tucker **[1]**. Another good background source is White **[2]**. Read and Corneil **[3]** review some of the deterministic approaches to the isomorphism problem.

A *graph* may have one or more self-loops at any vertex and arbitrarily many edges between any pair of vertices. If it has neither self-loops nor multiple edges, a graph is said to be *simplicial*. Also, every edge of a graph, including self-loops, has two *orientations*, that is, directions of possible traversal.

Graphs are taken to be connected and *surfaces* to be closed and oriented, unless the immediate context suggests otherwise. The orientable surface of genus k is denoted by S_k. *Imbeddings* have the *cellularity property* that the interior of every region is simply connected.

A *rotation* at a vertex v is a cyclic permutation of the edge-ends incident on v. Thus a d-valent vertex admits $(d-1)!$ rotations. A list of rotations, one for each vertex of the graph, is called a *rotation system*.

Any imbedding of a graph G in an oriented surface induces a rotation system. In particular, the rotation at vertex v is the cyclic permutation corresponding to the order in which the edge-ends are traversed in an orientation-preserving tour around v, as illustrated in Figure 1.

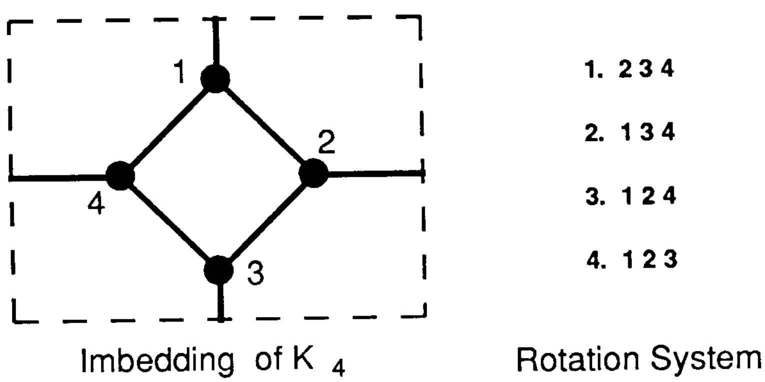

Figure 1: An imbedding of K_4 and the corresponding rotation system.

Conversely, by the Heffter-Edmonds principle, every rotation system induces a unique imbedding of G into an oriented surface. The bijectivity of this correspondence implies that the number of different ways to imbed a graph of valence sequence $d_1, \ldots, d_n$ into a closed oriented surface is:

$$\prod_{j=1}^{n} (d_j - 1)! \, .$$

For any graph G, if the number of imbeddings in the surface S_k is denoted by g_k, then the sequence

$$g_0, g_1, g_2, \ldots$$

is called the *genus distribution* of G. One might naively hope that the genus distribution is a complete invariant of isomorphism type, and that polynomial-sized random samples from their distributions would be adequate to distinguish two different graphs, perhaps subject to the proviso that they satisfy a connectivity condition.

Although two non-isomorphic trees with the same valence sequence have the same genus distribution, there already exist fast isomorphism tests for trees. Counterexamples for the 2-connected case were constructed by Gross *et al.* **[4]**; for the non-simplicial 3-connected case by McGeoch **[5]**; and for the simplicial 3-connected case by Rieper **[6]**. Moreover, McGeoch **[7]** found that the genus distributions of circular ladders and Möbius ladders are identical for genus larger than one; furthermore, their only difference at the low end is that the Möbius ladder has no spherical imbeddings and two more toroidal imbeddings than the circular ladder, which has two spherical imbeddings.

In jointly initiating this probabilistic strategy, Gross and Furst **[8]** anticipated the possibility of such obstacles, and they developed a partially-ordered hierarchy for topological invariants, in which progressively more information about imbeddings is retained as one ascends. An arbitrarily large (finite) complete invariant may retain its usefulness for isomorphism testing if highly informative samples are accessible.

Pragmatically, it might be sufficient for an invariant to be “nearly complete”, in the sense that only a tractably small number of domain objects are assigned the same invariant value, if there exist effective tests for distinguishing non-isomorphic objects that share an invariant value. With this in mind, Chen and Gross **[9]** have proved that the average genus is a candidate for a nearly complete invariant for 3-connected graphs, in a strong, statistically suggestive variation: The set of values of average genus realized over this class of graphs has no finite limit points. It seems reasonable to seek progressively stronger statistical invariants by ascending the hierarchy. *Stratified graphs* (see §2) are a characterization of imbedding systems that Gross and Tucker **[10]** have demonstrated to be a complete invariant of homeomorphism type.

Enumerative (see §3.1) and statistical investigations (see §3.2) are a growing interest in topological graph theory, with an independent natural expansion occurring in parallel to the present quest for a solution to the isomorphism problem. Deterministic algorithms (see §3.3) for topological properties are another co-pursuit.

2. Complete and Nearly-Complete Invariants

For the purpose of probabilistic isomorphism testing, a complete invariant may be unsatisfactory, unless the features that distinguish the invariant value for any given graph from the features of the "devil's best effort" at fooling us are feasibly computable. In this section, we review the provably complete invariant called a *stratified graph*, whose known distinguishing aspects seem to be difficult to calculate or to estimate, and a related complete invariant called an *adjacency hypercube*. Adjacency hypercubes are amenable to sample selection from well-controlled partitions, and they do an excellent job of distinguishing classical "devil's pairs". One strategic principle behind simultaneous consideration of two complete invariants, one with topologically tractable properties and the other with computationally convenient properties, is to derive a combined invariant that shares the desirable properties of both.

Two imbeddings of a graph G are regarded as *VM-adjacent* if one can be obtained from the other by moving one edge-end ("vertex modification") in the rotations at its vertex, and *EM-adjacent* if by moving both ends ("edge modification") at their respective vertices. The resulting combinatorial object is called the *stratified graph* for G, and is denoted by SG. The induced subgraph in SG on all G-imbeddings into S_j is called the j^{th} *stratum*, and is denoted by S_jG. The sequence of stratum sizes is simply the genus distribution.

For clarity, we refer to "vertices" and "edges" in G, and to "points" and "lines" in SG. Lines of SG that lie within a single stratum are called *level lines*, and the lines that run between consecutive strata are called *transverse lines* (or *transversals*). The set of transversals forms the Hasse diagram for a ranked poset on the imbeddings of G, in which the rank of a point equals the genus of the associated imbedding surface.

Figure 2 illustrates how examination of their first strata can quickly distinguish a circular ladder (left side) from a Möbius ladder (right side), even though the global difference between the two ladder graphs themselves is extremely small. This phenomenon is what first motivated their study.

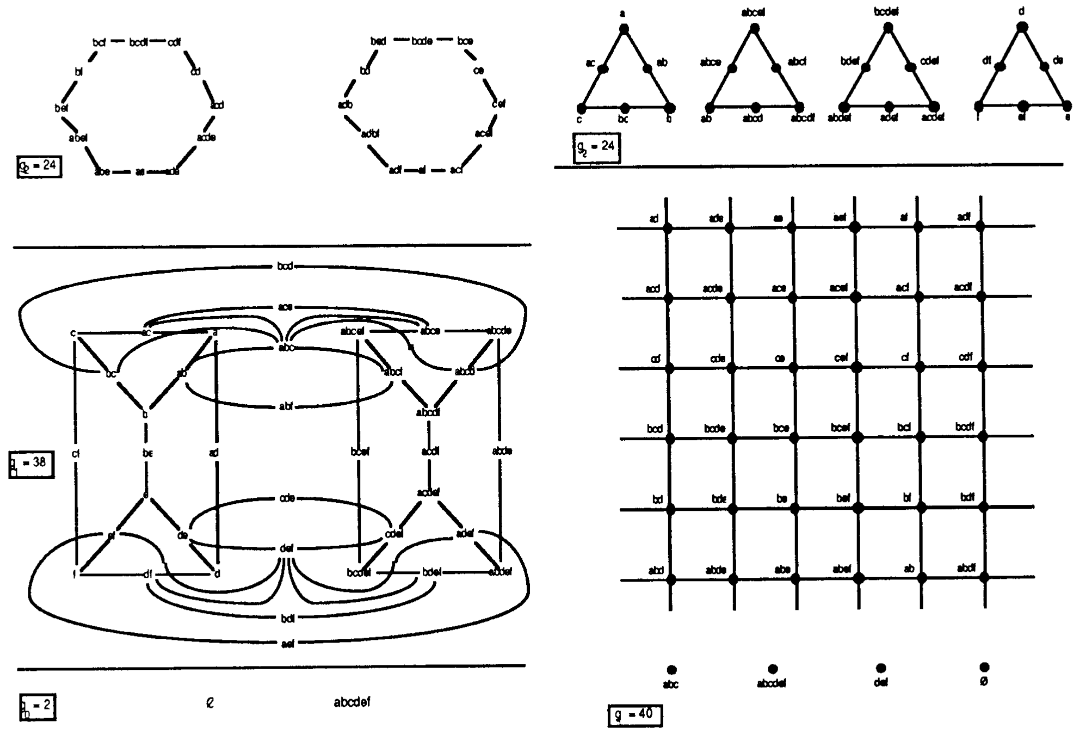

Figure 2: Highly similar graphs with overtly distinct strata.

Gross and Tucker **[10]** have proved that the graph *G* can be reconstructed from the link of any point of *SG*. The present form of the reconstruction depends on distinguishing the VM-lines from the EM-lines, which seems a trifle inelegant as abstract graph theory, but which offers no impediment to the objective of isomorphism testing. They also have proved that the VM-subgraph of *SG* is a Cartesian product of Cayley graphs that depend only on the valence sequence of *G*.

Stratified graphs also offer an insight into the contrasting results that, whereas the maximum-genus problem is solvable in polynomial time **[11]**, the minimum-genus problem is NP-complete **[12]**. In particular, Gross and Rieper **[13]** have proved that there are no strict local maxima to be encountered in an ascent toward maximum genus, yet there may exist arbitrarily deep, strict local minima.

Besides their possible use in isomorphism testing, stratified graphs have many properties of interest. For instance, there is an induced subgraph of the clique graph of the total link of any point of the stratified graph *SG* that is isomorphic to the *medial graph*. (See **[14]** for further discussion of medial graphs.)

3. Enumeration, Algorithms, and Computer Graphics

Beyond the central payoff of complete invariants and of a rapidly improving understanding of how to sample them, this program of research has inspired the growth of enumerative methods in topological graph theory, of statistical investigations, and of algorithmic research.

3.1 Enumerative Methods

The study of enumerative methods for topological graph theory was inaugurated by Gross and Furst **[8]**. Furst *et al*. **[15]** calculated the genus distributions for two infinite classes of graphs, known as closed-end ladders and cobblestone graphs. Gross *et al.* **[16]** used the counting formula of Jackson **[17]** to derive the genus distributions of bouquets.

In 1988, Mull *et al.* counted the congruence classes of imbeddings of wheels and complete graphs. Rieper **[18]** computed not only the genus distributions of dipoles, but also the region-size distributions of bouquets and dipoles. Rieper **[18]** was also first to study the relationship of imbedding distributions to Stirling numbers and to use Redfield enumeration for topological problems. Other results on imbedding distributions have been obtained by Lee and White (in 1989) and by Schwenk and White **[19]**.

Mohar **[20]** introduced the overlap matrix for the study of imbeddability, and Chen *et al.* **[21]** have used overlap matrices to count non-orientable imbeddings.

3.2 Statistical Results

The earliest statistical results about imbedding distributions for individual graphs are the proofs by Furst *et al.* **[15]** that closed-end ladders and cobblestone paths have strongly unimodal genus distributions. Subsequently, Gross *et al.* **[16]** proved that bouquets also have strongly unimodal genus distributions.

The average genus of an individual graph is another topic for study first formally proposed by Gross and Furst **[8]**, and which is now leading to interesting results of various kinds. For instance, Gross *et al.* **[4]** have calculated the first few positive values that the average genus may assume over the class of all graphs, and they proved that arbitrarily many 2-connected non-simplicial graphs may have the same average genus.

Chen and Gross **[22]** established a Kuratowski-type theorem for average genus, by characterizing the graphs with average genus one or larger in terms of the complement of a maximum cactus. (Of course, Nordhaus *et al*. **[23]** long ago obtained the analogous result for maximum genus.) Chen and Gross **[9]** proved that only finitely many 3-connected graphs have their average genus within any real finite interval, from which it follows that there are no limit points for the set of values of average genus of 3-connected graphs; they also proved that the number 1 is the smallest real limit point of the set of possible values of the average genus, and that there are no lower limit points.

Schwenk and White **[19]** have calculated the average genus of closed-end ladders and of cobblestone paths. Stahl **[24]** has studied the average genus of a class of graphs with the same number of edges.

3.3 Topological Algorithms

A variety of viewpoints are leading to outstanding results with major algorithmic implications for topological graph theory. For instance, Robertson and Seymour **[25]** have developed the theory of graph minors, and Thomassen **[12]** proved that the minimum genus graph problem is NP-complete.

By way of contrast to the NP-completeness of the minimum genus problem, Furst *et al.* **[11]** devised a polynomial-time algorithm for maximum genus. The maximum genus problem was originally formulated by Nordhaus *et al.* **[26]**. Although powerful characterizations of maximum genus have been obtained by Xuong **[27]**, by Homenko *et al.* **[28]**, and by Nebesky **[29]**, the fastest previously known algorithms required exponential time.

Gross and Rieper **[13]** have explained the contrast in the complexity of these two topological problems in terms of stratified graphs, by proving that whereas there are no strict local maxima in the stratified graph to serve as obstacles to hill-climbing toward the global maximum genus, there might exist arbitrarily deep local minima to impede an attempted descent toward the global minimum genus. Moreover, Gross and Rieper **[30][31]** demonstrate how to construct strict local minima in Hamiltonian graphs.

3.4 Graph Synthesis

The results of Chen and Gross **[32]** extend general results and methods of Whitney **[33]** and Tutte **[34]**. For instance, Chen and Gross prove that if H is a 3-connected simplicial graph that is homeomorphic to a subgraph of a 3-connected simplicial graph G, then G has a linear synthesis from H in the following sense. There is a sequence

$$H \approx G_0, G_1, \ldots, G_n \approx G$$

of 3-connected graphs such that for $j = 1, \ldots, n$, the graph G_j can be obtained from the graph G_{j-1} by adding one new edge e_j so that each endpoint of e_j is either a vertex or a midpoint of an edge of G_{j-1}.

Gross and Rieper **[30]** also derive a new general result on graph synthesis. In particular, they prove that the vertices of any graph can be partitioned into two parts so that the induced subgraph on each part has even valence at every vertex. They also give an efficient algorithm for constructing such a partition.

4. Summary

Stratified graphs are large, complete invariants of isomorphism type. There now exist powerful characterizations of the structure of stratified graphs. This probabilistic and topological approach to the graph isomorphism problem is also yielding interesting mathematical by-products. In particular, the pursuit of enumerative and related statistical results about the imbedding systems of graphs is an emerging focus of topological graph theory. Algorithmic results obtained under this approach have sharpened the understanding of maximum genus. Moreover, the study of limit points for average genus has motivated the derivation of new results in the classical area of graph synthesis.

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Received: Abstract May 5, 1990

Received: Manuscript August 7, 1990

Graph Theory Notes of New York XIX, 18–25 (1990)
New York Academy of Sciences.

[GTN XIX:2] EXTREMAL PROBLEMS FOR RANDOM WALKS ON GRAPHS

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Abstract
Random walks on graphs have generated a great deal of recent interest on account of their new applications to randomized algorithms in computer science, together with new progress in their analysis. Here we survey briefly an aspect which is of natural interest in graph theory: the question of extremal graphs and extremal trees for certain random walk parameters.
The parameters discussed here are hitting time (expected number of steps to reach a fixed vertex) and cover time (expected number of steps to hit all vertices), as a function of the total number of vertices of the graph or tree. As we shall see the problems range from easy to daunting and the results from intuitive to surprising.

1. Introduction

Let G be a connected graph on n vertices, and let v be a fixed vertex of G. A *random walk* on G, beginning at v, is a stochastic process whose state at any time t is given by a vertex of G; at time 0 it is at vertex v, and if at time t it is at vertex u, then at time $t+1$ it will be at one of the neighbors of u, each neighbor having been chosen with equal probability.

The random walk thus constitutes a Markov chain, with state transition probability $p_{x,y} = 0$ if y is not adjacent to x and $p_{x,y} = 1/d(x)$ if y is adjacent to x and x has degree $d(x)$. The Markov chain will be irreducible (unless G is bipartite) and it is easily verified that its stationary distribution π satisfies $\pi_x = d(x)/2m$, where m is the number of edges of G.

Thus, we have that in the limit, the probability of being at any particular vertex is proportional to its degree regardless of the structure of G. This remarkable fact is the key to numerous applications.

In Aleliunas *et al.* **[1]**, random walks are used to establish the existence of short universal sequences for traversing graphs; in Doyle and Snell **[2]** they are elegantly associated with electrical networks; in Borre and Meissl **[3]** they are employed to estimate measurements given by approximate differences. Dyer, Frieze and Kannan **[4]** made use of random walks on graphs to obtain the first randomized polynomial-time algorithm for estimating the volume of a convex body, and similar algorithms have now been obtained for computing the permanent (Jerrum and Sinclair **[5]**) and sorting with expensive comparisons (Karzanov and Khachiyan **[6]**). Recently, Coppersmith et al. **[7]** have found an application of random walks to on-line algorithms.

Aldous **[8]** gives many other contexts in which random walks on graphs arise, and a valuable bibliography **[9]** compiled by the same author lists numerous additional references on the subject. Random walks on trees, incidentally, are of independent interest (see, for example, **[10]**–**[16]**).

In what follows we will not dwell on applications, but will instead consider two parameters associated with random walks. Although these parameters do in fact arise in applications, our object will be to take the graph theorist's point of view and to try to determine the ranges of these parameters as effected by the structure of G.

2. The Parameters

The *hitting time* $H_G(x, y)$ from x to y is defined to be the expected number of steps for a random walk on G beginning at vertex x to reach vertex y for the first time. Thus, for example, if x and y are at opposite ends of a path on n vertices then we have a *standard* random walk with reflecting barrier, and any of a number of arguments shows that the hitting time from x to y is precisely $(n-1)^2$. Those arguments can be generalized easily enough to cover any case where there is a unique path from x to y; the following lemma is paraphrased from Moon **[15]** (Theorem 6.1, page 48).

Lemma (hitting time formula). Let vertices x and y be at distance k in a graph G, with a unique path $x = v_0, v_1, \ldots, v_k = y$ between them. For each i, $0 \le i \le k$, let G_i be the component of $G - \{v_{i-1}\} - \{v_{i+1}\}$ which contains the point v_i; similarly G_0 will be the component of $G - \{v_i\}$ containing x. Let m_i be the number of edges in G_i. Then the expected hitting time $H_G(x, y)$ is equal to

$$k^2 + 2\sum_{i=0}^{k-1} m_i(k-i) \qquad \blacksquare$$

Determining $H_G(x, y)$ when there is more than one path between x and y is not so straightforward, but still computable in principle. Chandra *et al.* **[17]** gave an elegant formula for the *commute time* $H_G(x, y) + H_G(y, x)$ in terms of the electrical resistance of a network corresponding to G, and Tetali **[18]** has now given an electrical interpretation for the hitting time; but neither characterization has yet been useful in extremal problems.

Closely related to hitting time is the *cover time* $C_G(x)$, defined to be the expected number of steps before a random walk beginning at x hits all the other vertices of G. Much attention has recently been focussed on the cover time: see for instance Aldous **[19]** and the five subsequent papers in that issue of the *Journal of Theoretical Probability*.

Clearly hitting time and cover time are related by $G_G(x) \geq H_G(x, y)$, but there is a less obvious relation in the opposite direction, found by Matthews **[20]**: if μ_G is the maximum over all pairs u, v of $H_G(u, v)$, then $C_G(x) \leq \mu_G log n$. This relation has been even further tightened by Zuckerman **[21]** to cover cases where most hitting times are small.

We are now ready to formulate $2^3 = 8$ extremal problems in one question, as follows:

> What is the maximum (or minimum) hitting time (or cover time) among all connected graphs (or all trees) on n vertices? □

3. The Results

Let us consider hitting time first; two of the problems can immediately be disposed of, for the minimum hitting time $H_G(x, y) = 1$ can be achieved on a graph or tree by having x be a vertex of degree 1 pendant to y. The maximum hitting time on a tree is also easily determined; since there all paths are unique the Lemma above may be applied, and the unique extremal case is given (as expected) by the endpoints of a path.

One might suspect at first that the path also maximizes hitting time over *graphs*, but in fact it has been known for many years that there are graphs G (barbells, for example) with vertices x, y such that $H_G(x, y)$ is $\Omega(n^3)$. An upper bound of n^3 was obtained by Lawler **[22]**, who attributed the problem of determining the extremal value to Paul Erdős; actually that bound (for cover time, hence for hitting time) had already been obtained by Alelunias *et al.* **[1]** as a critical part of their landmark paper on universal sequences. However, the value of the least constant c such that $H_G G(x, y)$ is bounded asymptotically by cn^3 remained unknown until very recently.

Finally, Brightwell and Winkler **[23]** were able to find the precise graph maximizing $H_G G(x, y)$, solving the problem completely. The graph is a *lollipop* consisting of a clique on $2n/3$ vertices with a path on the remaining $n/3$ vertices attached at one end; the start vertex x is in the clique, and the end vertex y is of course at the far end of the path. The hitting time turns out to be $\frac{4}{27}n^3$ plus lower order terms. Interestingly, the techniques of **[23]** are elementary; just graph theory and induction. The induction hypothesis is that for any nonnegative constant M, the maximum value of $H_G G(x, y)$ plus M times the number of edges is achieved by *some* graph of lollipop form.

We switch now to cover time, where much less is known. Oddly, we believe that the same lollipop graph described above also maximizes cover time, thus bounding the latter asymptotically also by $\frac{4}{27}n^3$. However, to prove this will require new techniques.

Embarrassingly, the *minimum* possible cover time for a graph is also unknown. Let us first consider the complete graph K_n, an obvious candidate for having the lowest cover time among all n-vertex graphs.

To compute cover time for K_n is to solve the elementary *coupon collector's problem*. For a given walk let T_i be the step number at which the i^{th} new vertex is reached; thus $T_1 = 0$, $T_2 = 1$ and T_n is the observed time to cover all the vertices of G. Since the probability of hitting a *new* vertex in a single step after i vertices have been hit is precisely $(n-i)/(n-1)$, we have that the expected value of $T_{i+1} - T_i$ is $(n-i)/(n-1)$; therefore

$$\begin{aligned} G_G(x) &= E(T_n) = \sum_{i=0}^{n-1} E(T_{i+1} - T_i) \\ &= (n-1)\sum_{i=1}^{n-1} 1/(1+i) \sim (n-1)\log n \end{aligned}$$

However, K_n is by no means the only graph with cover time as low as $n\log n$; for example the complete bipartite graph $K_{n/2,\, n/2}$ has nearly identical cover time, and even *expander* graphs with only linearly many edges can have cover times of that order.

Worse, one can actually beat the complete graph with certain carefully chosen graphs and starting vertices. Of these, we believe a graph we call the *bomb graph* will eventually prove to be extremal. This graph consists of a clique on about $n - \ln n$ vertices, about $\sqrt{n}$ of which are attached to a vertex v which begins a dangling path of length $\ln n$. At the end of the path, or *fuse*, is the start vertex x. Note that the bomb graph still has cover time of order $(1 - o(1))n\log n$, beating K_n by only a tiny amount.

If we restrict consideration to *regular* graphs then K_n probably has the smallest cover time, but, again, not by much. Aldous **[24]** has recently shown that no regular graph can have cover time less than a constant times $n \log n$. It also seems likely that K_n has the least cover time starting from a *random* vertex (stationary distribution); here again Aldous **[25]** has shown a lower bound of $cn \log n$.

Perhaps the simplest next step we could reasonably ask for is to show that the correct constant is 1, that is, that no graph (or no regular graph *etc.*,) can have $\lim(C_G(x)/n \log n) < 1$.

The cover time of the *star* S_n (with $n-1$ leaves) is easily seen by another coupon collector's analysis to be

$$2(n-2)\sum_{i=0}^{n-2} 1/(1+i) \sim 2(n-2)\log n$$

assuming we start from a leaf, since it takes two steps to move from leaf to leaf. Kahn *et al.*, **[12]** were able to show that no tree has a cover time which is asymptotically less than half this number, and then Devroye and Sbihi **[10]** showed a lower bound of $2n \ln n - O(n \ln \ln n)$ for the cover time of a tree. Thus the star cannot be beaten by much, but one might think that a tree-like version of the bomb graph could win by a nose.

Interestingly, this is not the case: Brightwell and Winkler **[26]** were able to show that the star is the precise extremal case, except for $n = 4$ where there is a tie. Again, their methods were elementary and inductive.

Why doesn't a version of the bomb graph apply here? The key seems to be that after travelling down the fuse in the original bomb graph and bouncing around the clique, when we hit the vertex which attaches the fuse we rarely re-enter the fuse. In a tree version the star-leaf which is attached to the fuse has degree only two, so that on half the occasions on which that vertex is reached the walk subsequently wastes time in the fuse.

If one end of an extremal problem for trees leads to a star, then of course we expect the other end to be a path. Indeed we are morally certain that the *greatest* cover time for a tree on n vertices is attained by a path, starting at a central vertex; that time would be about $\frac{5}{4}n^2$. Annoyingly, we have no proof. In **[26]** we offer a small consolation: a path, starting from an endpoint, does have the greatest possible *cover-and-return* time; that is, expected number of steps (here, $2(n-1)^2$) to cover all vertices and then return to the start.

But *surely*, someone out there can show that the path has greatest cover time among trees. Please!

Acknowledgements

The author is on leave from Emory University. This research at Emory is supported by ONR grant N00014 85-K-0769. A later version of this paper will be submitted to the Proceedings of the 21st Southeastern International Conference on Combinatorics, Graph Theory and Computing.

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Received: July 27, 1990

Graph Theory Notes of New York XIX, 26–30 (1990)
New York Academy of Sciences.

[GTN XIX:3] GRAPH PROPAGATORS

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Abstract

A graph propagator is defined by $I(G\backslash u)I(G\backslash v) - I(G)I(G\backslash u, v)$, where I is a graph invariant and u, v are vertices of a graph G. Some previously known properties of propagators are outlined. A few novel results are reported for the case when I is the number of independent vertex sets.

1. Introduction

Let u and v be two vertices of a graph G. By $G\backslash v$ we mean the subgraph of G obtained by deleting from G the vertex v. The subgraph $G\backslash u, v$ is obtained from G by deleting vertices u and v.

Let I be some graph invariant. Then the quantity γ_{uv}, defined by

$$\gamma_{uv} = \gamma(G;I) = I(G\backslash u)I(G\backslash v) - I(G)I(G\backslash u, v)$$

is called a *propagator*. More precisely, γ_{uv} is the propagator for the vertices u and v of the graph G, associated with the invariant I. One can consider γ_{uv} to be an element of a *propagator matrix*. It is easy to see that $\gamma_{uv} = \gamma_{vu}$.

The concept of graph propagators appears to have originated with Merrifield and Simmons **[1]**, who considered the special case when $I(G)$ is the number of sets of independent vertices of the graph G. (Note that for vertices that are adjacent, our definition of the propagator is somewhat different from that used in **[1]**. For nonadjacent vertices u and v, the present definition is equivalent to that of Merrifield and Simmons.) The propagator γ_{uv} can be interpreted as a measure of the interaction between the vertices u and v as far as the graph invariant I is concerned **[1]**. This interpretation is supported by the following elementary result.

Let $G_1 \cup G_2$ denote the disjoint union of G_1 and G_2. A graph invariant is called *multiplicative* provided

$$I(G_1 \cup G_2) = I(G_1)I(G_2)$$

Proposition 1. If I is a multiplicative graph invariant and u and v belong to distinct components of a (disconnected) graph G, then $\gamma_{uv} = 0$.

Proof: Let $u \in G_1$ and $v \in G_2$, the complement of G_1 in G. Then,

$$\begin{aligned}\gamma_{uv} &= I(G_1 \cup G_2 \backslash u)I(G_1 \cup G_2 \backslash \mathrm{v}) - I(G_1 \cup G_2)I(G_1 \cup G_2 \backslash u, v) \\ &= I(G_1 \backslash u)I(G_2)I(G_1)I(G_2 \backslash v) - I(G_1)I(G_2)I(G_1 \backslash u)I(G_2 \backslash v) = 0\end{aligned}$$

■

Two previously known properties of the characteristic **[2]** and the matching **[3][4]** polynomials can now be formulated in terms of propagators. We state them without proof, noting that Proposition 2 is just a graph-theoretical version of an old result in linear algebra **[5]**. Proposition 3 is based on a result discovered by Heilmann and Lieb **[6]**.

Proposition 2. If I denotes the characteristic polynomial of a graph then, for $u \neq v$,

$$\gamma_{uv} = \left(\sum_P I(G \backslash P)\right)^2$$

where P denotes a path and the summation is over all paths of G that connect vertices u and v. ■

Proposition 3. If I denotes the matching polynomial of a graph then, for $u \neq v$,

$$\gamma_{uv} = \left(\sum_P I(G \backslash P)\right)^2$$

with the same notation as in Proposition 2. ■

Recall that the *matching polynomial* of G is defined by

$$\sum_k (-1)^k m(G;k) x^{n-2k}$$

and that the *number of independent edge sets* is given by

$$\sum_k m(G;k)$$

where $m(G;k)$ is the number of k-element sets of independent edges in the graph G. Thus, we can deduce from Proposition 3 the following results.

Corollary 3.1. If I denotes the number of independent edge sets of a graph then, for $u \neq v$,

$$\gamma_{uv} = \sum_{P} (-1)^{|P|-1} \{I(G \backslash P)\}^2$$

where $|P|$ is the number of vertices in the path P. ■

Corollary 3.2. If I denotes the number of independent edge sets of a bipartite graph then, for $u \neq v$,

$$\gamma_{uv} = (-1)^{d(u,v)} \sum_{P} \{I(G \backslash P)\}^2$$

where $d(u, v)$ is the distance between the vertices u and v. ■

2. The Propagator Associated with the Number of Independent Vertex Sets

The number of independent vertex sets of a graph has been studied by several authors **[7]–[12]**. Merrifield and Simmons revealed remarkable applications of this quantity in chemistry **[1][11]**. From this point it is assumed that the propagator considered is associated with the number of independent vertex sets. Let this number be denoted by $S(G)$. Then the following relations hold **[7][12]**:

$$\begin{aligned} S(G_1 \cup G_2) &= S(G_1)S(G_2) \\ S(G) &= S(G \backslash v) + S(G \backslash N_v) \,. \end{aligned} \tag{1}$$

where N_v is the set consisting of vertex v and its adjacent vertices.

Lemma 4. If u and v are adjacent vertices then

$$S(G) = S(G \backslash u) + S(G \backslash v) - S(G \backslash u, v) \,. \tag{2}$$

Proof: Applying eqn(1) to $G \backslash u$ we obtain

(3) $S(G\backslash u) = S(G\backslash u, v) + S(G\backslash u, N_v)$.

Since u and v are adjacent, $u \in N_v$ and $G\backslash u, N_v = G\backslash N_v$. Consequently, $S(G\backslash u, N_v) = S(G\backslash N_v)$ and eqn(2) follows by combining eqn(1) with eqn(3). ■

Proposition 5. If u and v are adjacent vertices then

$$\gamma_{uv}(G) = S(G\backslash N_u)S(G\backslash N_v) .$$

Proof: Using Lemma 4 we easily transform $\gamma_{uv}(G)$ into

$$\{S(G) - S(G\backslash u)\}\{S(G) - S(G\backslash v)\} .$$

Proposition 5 follows from eqn(1). ■

Corollary 5.1. If u and v are adjacent vertices (and thus $d(u, v) = 1$), then

(4) $\mathrm{sign}(\gamma_{uv}) = (-1)^{d(u, v) + 1}$. ■

Conjecture. In the case of bipartite graphs, eqn(4) holds also for non adjacent vertices. □

Note that the above conjecture is precisely the same as a statement given in **[1]** (page144). Furthermore, examples show that eqn(4) is violated if u and v are nonadjacent in the case that G is not bipartite.

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Received: March 5, 1990

Graph Theory Notes of New York XIX, 31–34 (1990)
New York Academy of Sciences.

[GTN XIX:4] ON A CODING OF PLANE TREES AND ITS APPLICATIONS

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Let $\mathcal{F}_n$ denote the set of *rooted plane trees* with n vertices. The vertices of such trees are not labelled, although the root is distinguishable from other vertices. Two plane trees are considered the same if and only if they have the same ordered set of branches with respect to the root. The cardinality of $\mathcal{F}_n$ is (see for example, **[1]**)

$$|\mathcal{F}_n| = C_{n-1}$$

where $C_n = \binom{2n}{n}/(n+1)$ is the n^{th} Catalan number.

A *random plane tree* is a couple $(\mathcal{F}_n, P_n)$, where P_n is the uniform probability distribution (that is, for each tree $T \in \mathcal{F}_n$, $P_n = 1/|\mathcal{F}_n|$).

In this note we propose a way of coding plane trees analogous to that of Prüfer for labelled trees. In what follows we use a graphic representation of rooted trees in which the highest vertex is the root, the next level consists of the vertices incident to the root, the next lower level consists of the vertices incident to the vertices incident to the vertices in the previous level, and so on. Using this graphical representation, one can label vertices of plane tree $T \in \mathcal{F}_n$ as follows (see also the Figure)

(1) the root has the label 1,

(2) vertices are labelled by successive integers $2, 3, \ldots, n$ from left to right in the first level, then in the second level, and so on.

For each tree $T \in \mathcal{F}_n$ $(n \geq 3)$ labelling the vertices in this way, one can assign an $(n-2)$-dimensional vector $R_T = (r_1, r_2, \ldots, r_{n-2})$ such that r_i is the lowest label of the neighbors of the vertex $i+2$. For example, the tree in the Figure is assigned the vector:

$$R_T = (1, 2, 2, 2, 3, 3, 5, 5) .$$

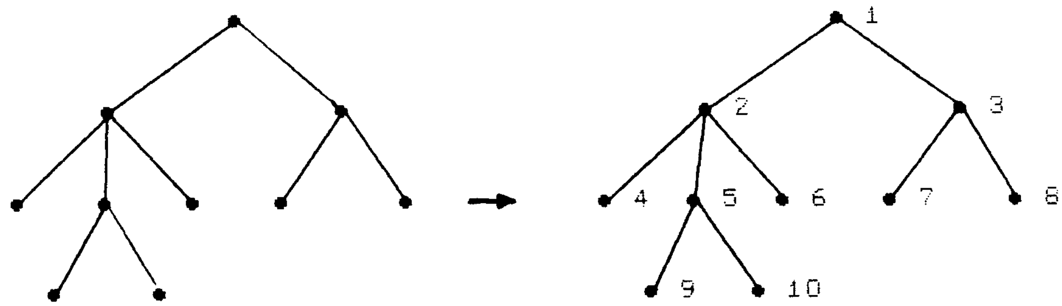

Figure: Labelling of a plane tree.

Generally, the vector R_T has the following properties

$$
(1)\qquad \begin{array}{l} 1 \le r_1 \le r_2 \le \ldots \le r_{n-2} \le n-1 \\ r_i \in \{1, 2, \ldots, i+1\} \text{ for } i = 1, 2, \ldots, n-2 \, . \end{array}
$$

The vertex v of the tree $T \in \mathcal{F}_n$ has degree i if and only if $i-1$ coefficients of the vector R_T are equal to v (as in the Prüfer code).

Fact. There exists a one-to-one correspondence between plane trees in the family $\mathcal{F}_n$ (where $n \ge 3$) and the family of $(n-2)$-dimensional vectors satisfying eqn(1). ■

This way of coding plane trees allows one to investigate properties connected with the degree of a fixed vertex.

Let $u_{n,k,i}$ denote the number of plane trees from family $\mathcal{F}_n$ such that the vertex with label k has degree i. From properties of plane trees, one can prove the following recurrence equations

$$
(2)\qquad \begin{array}{l} u_{n,1,1} = u_{n,k,2} = C_{n-2} \text{ for } n \ge 2,\ 1 \le k \le n-1, \\ r_1 \in \{1, 2, \ldots, i+1\} \text{ for } i = 1, 2, \ldots, n-2 \, . \end{array}
$$

have a solution of the form

$$
u_{n,1,i} = \frac{1}{n-1}\binom{2n-i-3}{n-2} \text{ for } n \ge 2,\ 2 \le i \le n-1 \, .
$$

Note that Dershowitz and Zaks **[2]**, and also Kemp **[3]** have found $u_{n,k,i}$ using a different method. For each plane tree $T \in \mathcal{F}_n$ such that the k^{th} vertex has degree i, vector R_T satisfies the following relations

$$r_1 \le r_2 \le \ldots \le r_j \le k-1$$
$$r_{j+1} = r_{j+2} = \ldots = r_{j+i-1} = k$$
$$k+1 \le r_{j+1} \le r_{j+i+1} \le \ldots \le r_{n-2}\ .$$

where $k-2 \le j \le n-i-1$. Using this fact, one can obtain the following recurrence relation:

(3) $$u_{n,k,i} = u_{n,k,i+1} = u_{n-1,k,i-1}$$

where $n \ge 3$, $1 \le k \le n-1$ and $1 \le i \le n-k+1$. For $k = 1$, eqn(3) is a special case of the system eqn(2).

Let $D_{n,k}$ denote the degree of the vertex with label k in a random plane tree $T \in \mathcal{F}_n$.

Theorem. For $n \ge 3$, $2 \le k \le n$ and $r = 1, 2, 3, \ldots$

$$E(D^r_{n,k}) = \frac{C_n}{C_{n-1}} \sum_{p=0}^{j} \binom{r}{j}\binom{j}{p}(-2)^{j-p} E(D^p_{n+1,k}) + (-1)^r \sum_{j=0}^{r-1} \quad . \qquad \blacksquare$$

Using this theorem one can find the first two moments of the random variable $D_{n,k}$.

Corollary. For $n \ge 3$, the following relations hold:

(a) $E(D_{n-1}) = 3 - 6/(n+1)$

(b) $E(D^2_{n,1}) = 13 - 6(11n+2)/(n+1)(n+2)$

and for $k \ge 2$

(c) $$E(D_{n,k}) = \sum_{j=k-1}^{n-1} C_{n-j-1} C_j / C_{n-1},$$

(d) $$E(D^2_{n,k}) = \left(2\sum_{j=k-1}^{n-1} C_{n-j} C_j - \sum_{j=k-1}^{n-1} C_{n-j-1} C_j\right) / C_{n-1}. \qquad \blacksquare$$

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Received: May 6, 1990

Graph Theory Notes of New York XIX, 35 (1990)
New York Academy of Sciences.

[GTN XIX:5] GLOBAL DENSITY OF SPARSE VERTEX-RAMSEY GRAPHS

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Given a graph with j edges and k vertices, call the ratio j/k the *density* of G. The *global density* of G is the density of its densest subgraph.

Given a graph G with at least one edge and an integer $r \geq 1$, we say that a graph F is *vertex-Ramsey* with respect to G and r provided every r-coloring of the vertices of F results in a monochromatic copy of G. In **[1]** we proved that the global density of a vertex-Ramsey graph is at least $rD(G)/2$, where $D(G)$ is the largest minimum degree of any subgraph of G. We also know that $rD(G)$ is an upper bound.

> **Problem.** For given G and r, determine the infimum global density over all vertex-Ramsey graphs with respect to G and r. ■

For complete graphs the answer is known. Indeed, applying the pigeon-hole principle to the complete graph on $(m-1)r+1$ vertices, we observe that it is vertex-Ramsey with respect to $K(m)$ and r, and its density coincides with the above lower bound and is equal to $(m-1)r/2$. In view of the fact that Ramsey graphs may be locally very sparse, it is a little surprising that the global density with respect to complete graphs cannot be smaller than the trivial upper bound coming from the ordinary pigeon-hole principle.

The simplest unknown case is when G is a path on 3 vertices and $r = 2$. We only know that the infimum lies in the closed interval $[4/3, 7/5]$.

References

[1] T. Łuczak, A. Ruciński and B. Voight; Ramsey properties of random graphs, —submitted.

Received: May 20, 1990

Graph Theory Notes of New York XIX, 36–37 (1990)
New York Academy of Sciences.

[GTN XIX:6] THE EULER NUMBER OF A GRAPH

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A graph is *Eulerian* if it has a closed walk (that is, a walk beginning and ending at the same vertex) which traverses each edge of the graph exactly once. A graph is Eulerian if and only if each vertex has even degree; that is, is an *even vertex*. If a graph has odd vertices, it must have an even number of them since the degree sum of a graph is even.

We define the *Euler number* of a graph as the minimum number of edges which must be added to the graph to make it Eulerian. If a graph is Eulerian, the Euler number is zero. Note that some graphs are *incurable* and have no Euler number. For example, complete graphs of even order are incurable.

> **Problem 1.** Are there other classes of graphs having no Euler number and do they have a "nice" characterization? □

The following theorem establishes a lower bound for the Euler number of a graph. Call the Euler number of an incurable graph infinity.

> **Theorem 1.** If a graph G has $2k$ odd vertices, then its Euler number is at least k.

Proof: To render G Eulerian, the degree of each odd vertex must be increased by at least one, requiring k edges since G has $2k$ odd vertices. ■

> **Corollary 1a.** Let G have $2k$ odd vertices which form an independent set. Then the Euler number of G is k. ■

An examination of the ladder $P_n \times K_2$, with $n \geq 4$, shows that the Euler number of a graph can be k even though the odd vertices are not independent. Observe that, when $n = 3$, the ladder has Euler number three, while it has only two odd vertices. We pose the following problems.

Problem 2. Characterize those graphs with Euler numbers equal to k, and those with Euler number greater than k, respectively. □

Problem 3. What is the relationship (if any) between the Euler number of $U \times V$ and the Euler numbers of U and V? □

Received: June 14, 1990

Graph Theory Notes of New York XIX, 38–39 (1990)
New York Academy of Sciences.

[GTN XIX:7] WHAT IS THE MINIMUM LABEL NUMBER OF $U \times V$?

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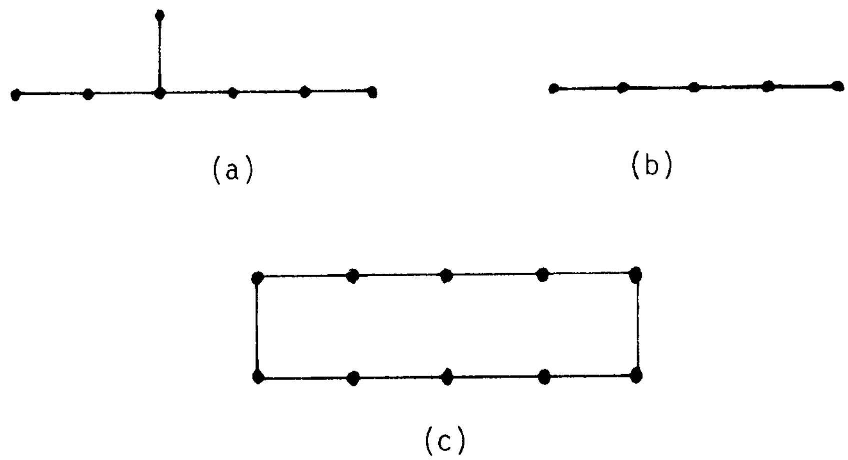
The *minimum label number* (MLN) of a graph G is defined in **[1]** to be the minimum number of vertices which must be labelled so that each vertex of G is distinguishable. The graphs of the Figure have minimum label numbers 0, 1, and 2, respectively. It should be noted that care must be used in selecting the vertices to be labelled.

It is shown in **[1]** that the MLN of a tree is at most one less that the number of end-vertices. Furthermore, the MLN of the hypercube Q_n is n.

The reader is reminded that the *Cartesian product* $U \times V$ of graphs U and V, with vertex sets $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$, respectively, has vertex set $\{(i, j) | (1 \le j \le n)\}$ and (i, j) is adjacent to (h, k) if and only if either (1) $i = h$ and v_j is adjacent to v_k in V, or (2) $j = k$ and u_i is adjacent to u_h in U. The set of vertices (i, j) *with i fixed* and $1 \le j \le n$ n is called the i^{th} V-copy, with U-copies defined similarly.

The *degree* of the i^{th} V-copy is the degree of u_i in U. The *distance* between the i^{th} and h^{th} V-copies is $d_u(u_i, u_h)$. In particular, they are *adjacent* whenever u_i and u_h are adjacent in U. The same applies to U-copies. Given the MLNs of U and V, we ask for the MLN of $U \times V$ or provide an upper bound.



Figure

Observe, to start, that if all of the vertices of a U-copy in $U \times V$ are labelled, then all the U-copies can be recognized as such. Now suppose that U and V have minimum label numbers r and s, respectively. Then a labelling of $U \times V$ can be obtained by labelling the m vertices of each of s U-copies corresponding to a labelled vertex of V (under some minimum labelling for V), or *vice versa*. We have, then, proved the following:

Theorem. Given graphs U and V on m and n vertices, and having MLNs r and s, respectively. Then $\mathrm{MLN}(U \times V) \leq \min(rn, sm)$. ■

Since $\mathrm{MLN}(Q_4) = 4$ and $Q_4 = C_4 \times C_4$, the upper bound of this theorem is probably far too large, for it yields 8 in this case (the MLN of a cycle is clearly two). We believe that a better bound is given by

$$\min(n + r - 1, m + s - 1)$$

since only one U-copy (V-copy) need be fully labelled, while the remaining $s - 1$ ($r - 1$) copies may be distinguished by labelling one vertex per copy (these all corresponding to the same vertex in U (V)).

References

[1] A. Fox; Minimum label numbers of graphs, *J. Undergraduate Mathematics*, —to appear.

Received: June 14, 1990

INFORMATION CORNER

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The Information Corner includes announcements about upcoming meetings, visiting faculty, and other information concerning activities of graph theoretical interest. Please suggest items for inclusion in the Information Corner of future issues of *Graph Theory Notes*.

I. Future Meetings of Interest to Graph Theorists

1990

December 3–7, 1990
1990 Austalasian Conference on Combinatorics
Palmerstown North, New Zealand

CONTACT: C.H.C. Little
Department of Mathematics & Statistics
Massey University
Palmerstown North, NEW ZEALAND

1991

January 7–10, 1991
Sixth Caribbean Conference on Combinatorics & Computing
The University of the West Indies
St. Augustine, Trinidad, W.I.

CONTACT: E. J. Farrell
Department of Mathematics
The University of the West Indies
St. Augustine, TRINIDAD, W.I.
FAX: (809)662-4414

January 14–15, 1991
AMS Short Course on "Probabilistic Combinatorics and its Applications"
San Francisco, California

CONTACT: D. Plante
AMS P.O. Box 6248
Providence, RI 02940

January 28–30, 1991
Second ACM-SIAM Symposium on Discrete Algorithms
San Francisco, California

CONTACT: SIAM Conference Coordinator
Dept. CC0590
3600 University City Science Ctr.
Philadelphia, PA 19104-2688
tel: (215) 382-9800
Fax: (215) 386-7999
email: siamconfs@wharton.upenn.edu

II. Ongoing Seminar Series in New York Area

Mondays at 6:30 p.m.
Seminar in Combinatorial Computing
CUNY Graduate Center

CONTACT: M. Anshel or G.S. Bloom, Computer Science Department
City College, CUNY
New York, NY 10031
e-mail: gsbcc@cunyvm.cuny.edu

Tuesdays at 1:30 p.m.
Discrete Mathematics & Operations Research Seminars
RUTCOR, Rutgers University

CONTACT: Peter Hammer
RUTCOR, Hill Center
Rutgers University
New Brunswick, NJ 08903

Tuesdays at 6:15 p.m.
Geometry Seminar
Courant Institute of Mathematical Sciences

CONTACT: R. Pollack, Mathematics Department
CIMS, New York University
New York, NY 10012

Wednesdays at 4:00 p.m.
Discrete Mathematics Seminar
Pace University

CONTACT: J.W. Kennedy or L.V. Quintas
Mathematics Department
Pace University
New York, NY 10038
e-mail: nfw2@pace.bitnet

Various Times
Discrete Mathematics Research Group Seminars
Bell Communications Research

CONTACT: C. Monma, BELCORE
435 South Street (2Q-346)
Morristown, NJ 07960

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and

(4) Contributions accepted directly by the Editors.

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Submit contributions for *Graph Theory Notes* to the Editors following the format of this issue.

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