MINIMUM LABEL NUMBERS OF GRAPHS

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I. INTRODUCTION. Some graphs contain vertices which are distinguishable by virtue of certain properties including degree, eccentricity, and distance relationships with other vertices (such as adjacencies). Such vertices are identifiable without labels. In the graph of Figure 1, the endvertex requires no label, nor do the vertices at distance 1 and 2 from it, respectively. The vertex of degree 4 is also uniquely identified; hence, the vertex of degree 2 at distance 3 from the endvertex is also identifiable. However, observe that the two remaining vertices, x and y, are entirely indistinguishable from one another, and require that one of them be labelled.

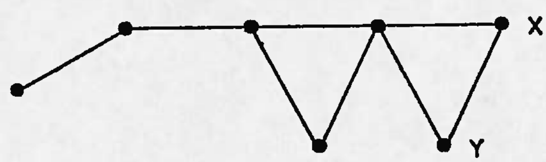

Figure 1

With this as out motivating idea, we define the minimum label number (MLN) of a graph as the smallest number of vertices which must be labelled so that each vertex of the graph is identifiable (possibly with reference to the labelled vertices). We use the notation of [1].

In this paper, we shall determine the MLN for various classes of graphs.

We conclude this section with several examples, the first of which is a graph with an MLN of zero. See Figure 2.

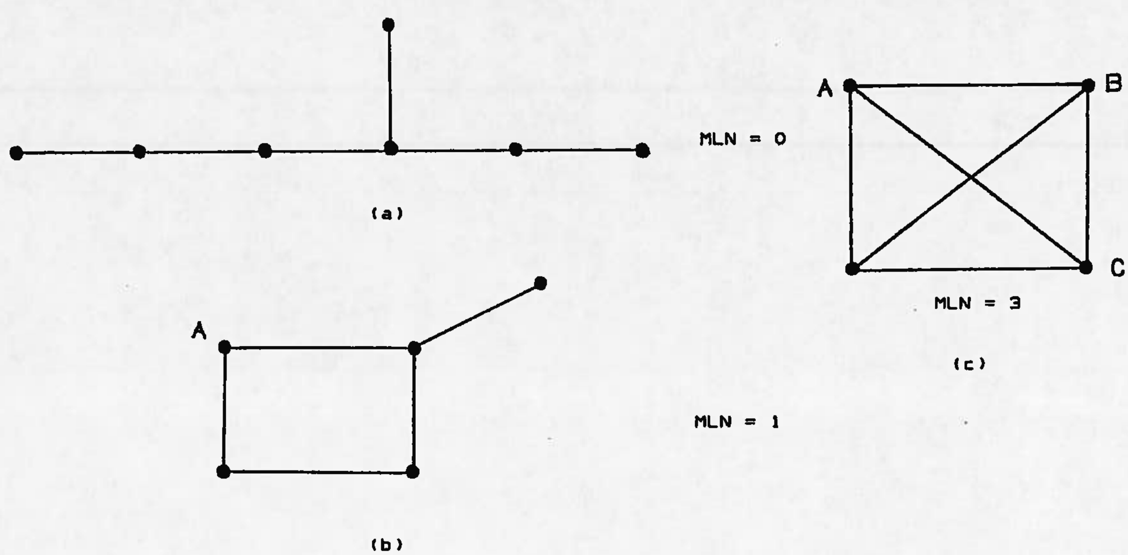

Figure 2

II. GRAPHS WITH SMALL MLN'S.

THEOREM 1. *The* MLN *of the path* P_n *is* 1.

PROOF. Upon labelling an endvertex, each vertex of the path is uniquely identifiable by its distance from the labelled vertex.

THEOREM 2. *The* MLN *of the cycle* C_n *is* 2.

PROOF. Label any pair of adjacent vertices x and y. Then every vertex of the cycle is uniquely determined by its ordered pair of distances from x and y.

The reader is reminded that the "wheel" on n vertices, W_n, is obtained by joining a single vertex to each vertex of the cycle C_{n-1}. Figure 3 shows W_5.

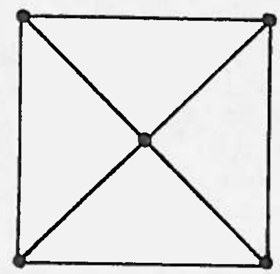

Figure 3

THEOREM 3. *The* MLN *of* W_n, $n \leq 5$, *is* 2.

PROOF. The center vertex is distinguishable, as its degree is $n-1$, while all other vertices have degree 3. Now label two adjacent vertices of degree 3, x and y. Then any other vertex of degree 3 is identifiable by its ordered pair of distances from x and y in the graph resulting from deleting the center vertex.

A starlike graph is a tree with only one vertex of degree strictly greater than two. This vertex is called the junction. The paths resulting from deletion of the junction are called *branches*.

THEOREM 4. *Let G be a starlike tree all of whose branches have different lengths. Then the* MLN *of* $G = 0$.

PROOF. The junction requires no label, nor do the endvertices, since each one has a unique distance from the junction. Any other vertex has a unique distance from the endvertex of the branch on which it lies.

III. MLN'S OF TREES. A star on n vertices, S_n, is a graph with only one vertex of degree strictly greater than one. Figure 4 depicts S_7.

THEOREM 5. *The* MLN *of* S_n *is* $n-2$.

PROOF. The center is the only distinguishable vertex, while the remaining $n-1$ vertices clearly require $n-2$ labels, since one of them is identifiable by its not being labelled.

A starlike tree with k branches such that each branch has the same length is called a k-equibranched starlike tree. We then have the following theorem:

THEOREM 6. *The* MLN *of a k-equibranched starlike tree is* $k-1$.

PROOF. Label $k-1$ of the endvertices, and proceed as in the proof of Theorem 4.

REMARK. Given a starlike tree G, with s_1 branches of length L_1, s_2 branches of length $L_2, \ldots$, and s_3 branches of length L_3, where the L's are all unique, then by an argument similar to that of Theorems 4 and

6, the MLN is

$$\left(\sum_{i=1}^{j} s_i\right) - j.$$

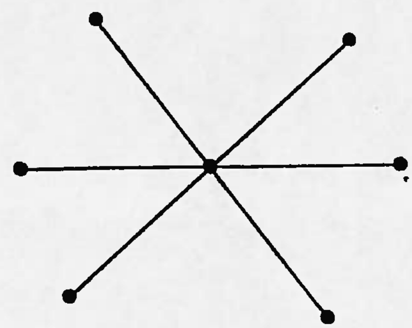

Figure 4

We now present an upper bound for the MLN of any tree.

THEOREM 7. *Given a tree T with s endvertices, then* $\mathrm{MLN}(T) \leq s - 1$.

PROOF. We proceed using induction on r, the radius. When $r = 1$, the tree is a star, and the theorem follows by Theorem 5. Assume the theorem is true for $r < r_0$.

Given a tree T with radius $r_0 + 1$, label all but 1 endvertex. Let S be the set of all vertices of T which are adjacent to any endvertex. Then all the vertices of S are distinguishable (since no endvertex can have more than one neighbor). Now let T' be the tree from T by deleting its endvertices.

Note that the radius of T' is r_0, and that its endvertices are exactly the members of S. It follows, by the inductive hypothesis, that all the vertices of T' are identifiable. Hence S_0 are the vertices of T. Since the cardinality of S is less than or equal to the number of endvertices of T, the theorem follows.

IV. HYPERCUBES. The hypercube Q_n is defined recursively. Q_1 is a copy of K_2. We obtain Q_2 by taking two copies of Q_1 and connecting corresponding adjacent vertices as shown in Figure 5.

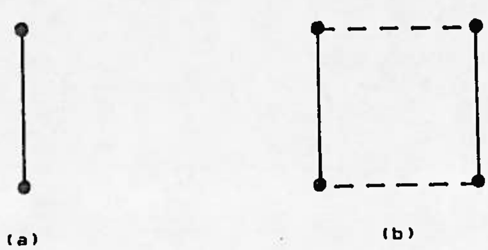

Figure 5

Observe that Q_3, defined similarly by connecting corresponding vertices of two copies of Q_2, is the highest order hypercube which is planar. Figure 6 depicts a planar drawing of Q_3.

Notice in Figure 5(b), that there are two ways to divide Q_2 into two copies of Q_1. The dashed lines could be thought of as the two copies of Q_1 with the solid lines as their connecting edges, or vice-versa.

We can analogously find three different ways to divide Q_3. By placing the second copy of Q_2 inside the first copy, we can easily show divisions along the vertical, horizontal and diagonal lines of Q_3. See Figure 7.

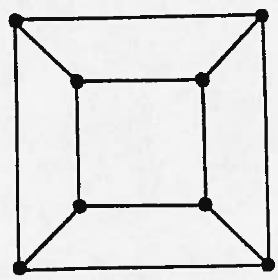

Figure 6

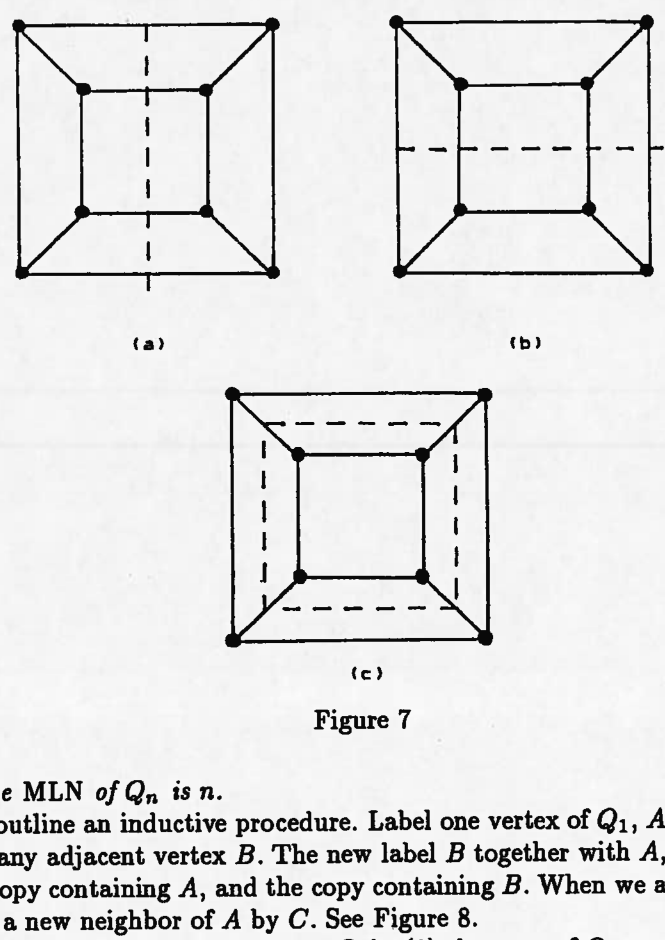

Figure 7

THEOREM 8. *The* MLN *of* Q_n *is* n.

PROOF. We shall outline an inductive procedure. Label one vertex of Q_1, A. In advancing to Q_2, retain the label A, and label any adjacent vertex B. The new label B together with A, define a division of Q_2 into two copies of Q_1, the copy containing A, and the copy containing B. When we advance to Q_3, we retain our Q_2 labelling and label a new neighbor of A by C. See Figure 8.

Observe that C effects a division of Q_3 into two Q_2's: (1) the copy of Q_2 containing A, and (2) the copy of Q_2 containing C.

This labelling is adequate since A and B suffice for the previous copy of Q_2, while vertices in the C copy (or new copy) are identifiable through the vertices of the old copy to which they correspond. This process is extendable for all n, in which case we see that n labelled vertices suffice.

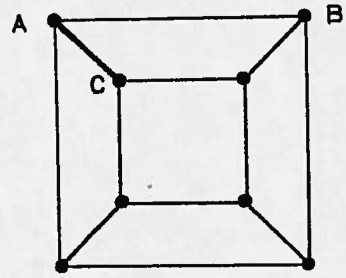

Figure 8

To show that n vertices are required, recall that there are n ways to divide Q_n into two copies of Q_{n-1}. Each division is accomplished by deleting 2^{n-1} edges comprising a "dimension set." Given $n-1$ or fewer vertices, not all divisions can be specified, thus making it impossible to distinguish all vertices.

The Ternary cube T_n is defined by $T_1 = K_3$ and $T_n = T_{n-1} \times K_3$. Figure 9 shows T_1 and T_2.

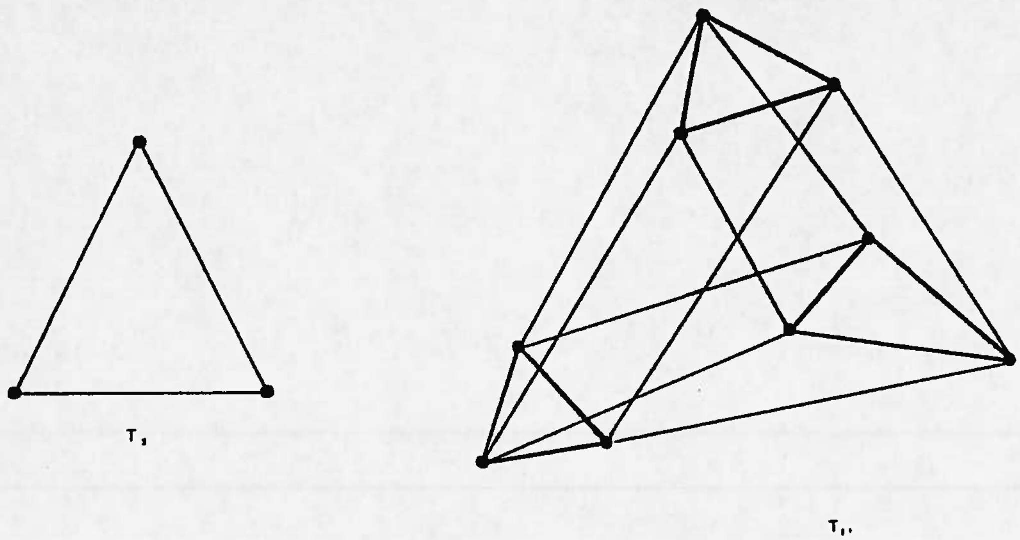

Figure 9

The MLN of T_n is clearly bounded above by $2n$, since T_1 has an MLN of 2, and if T_n has an MLN of $2n$ then 2 more vertices in T_{n+1} determine, together with $2n$ labelled vertices of one copy of T_n, three copies of T_n and hence determine all the vertices of T_{n+1}. We ask if in fact this is the MLN of T_n.

REFERENCES

[1] Lewinter, Martin, *Graphy Theory*. Vol. 13 Monographs in Undergraduate Mathematics, Greensboro, N.C.: Journal of Undergraduate Mathematics, Guilford College.

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